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OPTIMAL SYNTHESIS IN A TWO-DIMENSIONAL PROBLEM WITH ASYMMETRIC CONSTRAINTS ON THE CONTROL†

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A two-dimensional optimal control problem is considered on the assumption that the terminal time of the process is not fixed and the integral objective functional depends on a parameter. Asymmetric constraints are imposed on the control parameter. Two cases are considered: constraints of the same sign and constraints of different signs. In the case of constraints of different signs, if the parameters of the problem satisfy certain relations, one obtains chattering control, alternating with a control with two switchings and a first-order singular are when these relations are violated. In the case of sign-definite control the controllability domain is part of the plane bounded by two semiparabolas. Three types of control law are then possible, in two of which the system will hit the boundary of the controllability domain and move along it, while the third features a first-order singular are. As the parameter of the problem is varied, the phase portrait undergoes evolution and one of these three types is interchanged with another. The optimality of these control laws is rigorously established using a dynamic programming method. © 2003 Elsevier Science Ltd. All rights reserved.

Chattering control [1] is characterized by the property of the control parameter to switch values a renumerable number of times in a finite time interval, with the switching times accumulating at a certain point. The chattering phenomenon was first observed [2] when investigating the problem of suppressing noise in electronic devices. The problem was reduced to that of minimizing a functional which had the meaning of the integral square deviation and possessed a certain symmetry [2, 3].

In some cases such symmetry may not other. Over the period that has elapsed since the publication of Fuller's classical problem various modifications have been considered, such as Marshall's problem and the three-dimensional Fuller problem. In Marshall's problem, in particular, asymmetric constraints are imposed on the control parameter. In the case considered here the asymmetry also appears in the functional, which has the meaning of the integral square deviation evaluated with a certain weight. The essentially new fact in the problem with an asymmetric functional is the appearance of first-order singular control.

1. FORMULATION OF THE PROBLEM

A control system is given by equations of motion, constraints, and initial and terminal conditions

$$\begin{gathered} \dot{x} = y, \quad \dot{y} = u, \quad 0 \le t \le T, \quad a \le u \le b \\ x(0) = x^0, \quad y(0) = y^0, \quad x(T) = 0, \quad y(T) = 0 \end{gathered} \tag{1.1}$$

where T is the terminal time of the process, which is not fixed; u is a scalar control parameter and a and b are its limiting values.

The following functional is defined for motions of system (1.1)

$$J[u] = \int_0^T x^2(t)[Lu(t) + 1]dt \tag{1.2}$$

where L is a real parameter subject to the following constraints, depending on the signs of a and b

$$ab < 0: -1/b \le L \le -1/a; \quad a > 0: L \ge -1/b; \quad b < 0: L \le -1/a \tag{1.3}$$

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If $L = 0$, functional (1.2) is the mean square deviation. If $L \neq 0$, the functional has the meaning of a mean square deviation evaluated with a certain weighting function $h(u)$ which is linear in u.

The admissible controls are functions $u(t)$, integrable in any interval $[0, \delta]$, which satisfy the constraints in (1.1). The problem considered is to minimize functional (1.2) in the class of admissible controls and the corresponding motions of system (1,1). Constraints (1.3) imposed on the parameter L guarantee that the functional will be positive semidefinite.

If $L = 0$ in (1.2) and $b = -a = 1$ in (1.1), we have Fuller's problem, which has already been investigated [2, 3]. The problem with $L \neq 0$ and $b = -a = 1$ has also been investigated [4], and it has been shown that in that case the switching curve becomes asymmetric about the origin. It will be shown below, in particular, that asymmetry of the constraints imposed on the control implies asymmetry of the switching curve, even when $L = 0$.

2. THE MAXIMUM PRINCIPLE

The preliminary analysis of the problem will be based on the Maximum Principle. We shall assume that the conjugate variables p and q are taken with opposite sign, so that the conventional notation system of dynamic programming can be retained. Thus, $p = -\phi$, $q = -\psi$, where ϕ and ψ are the conjugate variables of the Maximum principle.

The Hamiltonian and its extremum values have the form

$$H(x, y, p, q) = py + qu + x^2(Lu + 1)$$

$$\min_u H = \min[F^a, F^b] = py + x^2 + (q + Lx^2)u^* \tag{2.1}$$

$$F^c = py + x^2 + c(q + Lx^2), \quad u^* = \frac{a+b}{2} + \frac{a-b}{2}\operatorname{sign}(q + Lx^2)$$

where, as always in this paper, $c = a, b$; the maximization operation has been replaced by minimization because of the reversal of the sign of the variables p and q; u^* is an optimal control.

It is evident from formulae (2.1) that the control is a bang-bang control, and therefore the solution of the problem is characterized by a switching curve (SC) which separates domains N^a and N^b in which the control takes values $u = a$ and $u = b$, respectively.

The following equality must hold on the branches of the switching curve

$$q + Lx^2 = 0 \tag{2.2}$$

It is then necessary that the expression $q + Lx^2$, as a function of time, should not vanish identically in any subinterval (t_1, t_2) of the interval $(0, T)$. Otherwise, as will be shown below, one obtains a control with a first-order singular arc.

Two equalities are valid identically with respect to time in the singular region: (2.2) and

$$py + x^2 = 0 \tag{2.3}$$

Equality (2.3) follows from (2.2) and the fact that the Hamiltonian vanishes on an optimal trajectory: $H(t) = H(x(t), y(t), u(t)) \equiv 0$.

Differentiating equality (2.2) along solutions of the Hamiltonian system

$$\dot{x} = H_p = y, \quad \dot{y} = H_q = u, \quad \dot{p} = -H_x = -2x(Lu + 1), \quad \dot{q} = -H_y = -p \tag{2.4}$$

we obtain

$$-p + 2xyL = 0 \tag{2.5}$$

Together with (2.3), this leads to the following equation (of a parabola) for the singular arc

$$x = -2Ly^2 \tag{2.6}$$

Differentiating equality (2.5) along trajectories of system (2.4) and using Eq. (2.6), we obtain an equality defining the singular control:

$$u^s = -\frac{1}{4}L \tag{2.7}$$

It follows from this equality that the singular arc is the part of the parabola (2.16) on which the relation $Ly > 0$ holds, since the other part of the parabola "recedes" from zero.

Using Hamiltonian (2.1) and singular control (2.7), one can show that Kelley's condition [5] is satisfied on the singular arc (2.6)

$$\frac{\partial}{\partial u}\frac{\partial^2}{\partial t^2}\frac{\partial H}{\partial u} = -8L^2y^2 \le 0$$

The reverse sign of the equality is due to the reversal of the sign of the conjugate variables.

3. DYNAMIC PROGRAMMING

Let $V(x, y)$ denote the optimal result function (Bellman's function) of problem (1.1), (1.2), that is, the minimum value of functional (1.2) on trajectories of system (1.1) that begin at the point (x, y). At all interior points of the controllability domain, the function $V(x, y)$ will satisfy the equation

$$yV_x + x^2 + (V_y + Lx^2)u^* = 0, \quad u^* = \frac{a+b}{2} + \frac{a-b}{2}\operatorname{sign}(V_y + Lx^2) \tag{3.1}$$

In the domains N^a and N^b Bellman's function satisfies the equations

$$F^a(x, y, V_x, V_y) = 0, \quad (x, y) \in N^a, \quad V_y + Lx^2 > 0, \quad u^* = a \tag{3.2}$$

$$F^b(x, y, V_x, V_y) = 0, \quad (x, y) \in N^b, \quad V_y + Lx^2 < 0, \quad u^* = b \tag{3.3}$$

where

$$F^c(x, y, V_x, V_y) = yV_x + cV_y + cLx^2 + x^2$$

Let $V^c(x, y)$ denote the restriction of Bellman's function to the domain N^c

$$V^c(x, y) = V(x, y), \quad (x, y) \in N^c \tag{3.4}$$

Thus, the function $V^a(x, y)$ ($V^b(x, y)$) satisfies Eq. (3.2) (Eq. (3.3)).

We need boundary conditions for Eqs (3.1)–(3.3). The terminal conditions in (1.1) yield the following value of Bellman's function at the origin

$$V(0, 0) = 0 \tag{3.5}$$

This equality will be treated as the boundary condition for Eqs (3.1)–(3.3). Generally speaking, boundary conditions should be specified on a curve, rather than at a single point. However, certain properties of Bellman's function enable us to derive a unique solution for which condition (3.5) is automatically satisfied.

The solution of problem (3.1)–(3.5) will be sought in the class of continuously differentiable functions. The aforementioned properties of Bellman's function are invariant in a group-theoretic sense. It is readily verified that the equations of motion and constraints (1.1) are invariant under to the transformation group

$$\bar{x} = \mu^2 x, \quad \bar{y} = \mu y, \quad \bar{t} = \mu t, \quad \bar{u} = u, \quad \mu > 0 \tag{3.6}$$

where μ is a scalar parameter. The meaning of this invariance is as follows. If $(x(t), y(t), u(t))$ is a solution of system (1.1) with initial point (x^0, y^0), then the triple $(\mu^2 x(t/\mu), \mu y(t/\mu), y(t/\mu))$ is a solution of the same system with initial point $(\mu^2 x^0, \mu y^0)$. In this situation a factor μ^5 will appear in functional (1.2), that is

$$V(\mu^2 x, \mu y) = \mu^5 V(x, y) \tag{3.7}$$

Differentiating equality (3.7) with respect to the parameter μ and then putting $\mu = 1$, we obtain the following equation, which must be satisfied by Bellman's function

$$2xV_x + yV_y - 5V = 0 \tag{3.8}$$

The general solution of Eq. (3.8) may be represented in the form

$$V(x, y) = y^5 \varphi(xy^{-2}) \tag{3.9}$$

The branches of the function $\varphi(z)$ corresponding to the branches $V^c(x, y)$ of Bellman's function according to (3.9) will be denoted by φ^c

$$V^c(x, y) = y^5 \varphi^c(xy^{-2}), \quad (x, y) \in N^c \tag{3.10}$$

Substituting expressions (3.10) into Eqs (2.2) and (2.3), we obtain ordinary differential equations for the function φ^c ($z = xy^{-2}$)

$$\varphi'(z)(1 - 2cz) + 5c\varphi(z) + z^2(1 + cL) = 0 \tag{3.11}$$

The general solutions of Eqs (3.11) are

$$\varphi^c(z) = A_c\left|z - \frac{1}{2c}\right|^{5/2} - \left(\frac{1}{c} + L\right)\left(z^2 - \frac{2}{3c}z + \frac{2}{15c^2}\right) \tag{3.12}$$

where A_c are constants of integration. It is assumed that the points $z = 1/(2c)$ are not in the interval in which the solution $\varphi^c(z)$ is defined. They are singular point of Eqs (3.11): The coefficient of the highest-order derivative vanishes at these points. If such a point is inside the interval in which the equation is defined, the general solution will depend on two parameters. For example, on different sides of the point $z = 1/(2a)$ one should use different constants A_a, say A_a and A_a^*. When that is done, the two branches corresponding to these two constants meet smoothly at $z = 1/(2a)$, forming a continuously differentiable function. This situation will arise below when constraints of different signs are considered.

Using relations (3.10) and (3.12), we obtain the following expressions for the values V^c of the function V in the domains N^c

$$V(x, y) = V^c(x, y) = A_c\left|\frac{1}{2c}y^2 - x\right|^{5/2} - \left(\frac{1}{c} + L\right)\left(x^2y - \frac{2}{3c}xy^3 + \frac{2}{15c^2}y^5\right), \; (x, y) \in N^c \tag{3.13}$$

Generally speaking, for each of the subdomains $y > 0$ and $y < 0$ of the domain N^c one should here use a different constant A_c. However, the condition that the function $V^c(x, y)$ be continuous at $y = 0$ implies that the constant must have a common value.

Thus, construction of the smooth function $V(x, y)$ reduces to determining the constants V_c.

The cases in which the signs of the parameters a and b are different or the same will be considered separately. This is because in the case $ab < 0$ the controllability domain (the domain from whose points admissible controls will steer the system to the origin) is the entire (x, y) plane, while if $ab > 0$, the controllability domain, as will be shown below, is only a part of the (x, y) plane enclosed between two semiparabolas.

4. CONSTRAINTS OF DIFFERENT SIGNS

Let us consider the case in which the constraints on the control have different signs, i.e. we wish to minimize the family of functionals (1.2) for motions of system (1.1) when the parameters satisfy the following relations

$$a \cdot b < 0, \quad -1/b \le L \le -1/a \tag{4.1}$$

Because of the constraints in (1.1) on the control, a singular arc (2.6) will be obtained in this case only for the following values of the parameter L

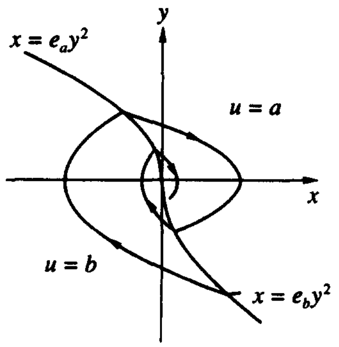

Fig. 1

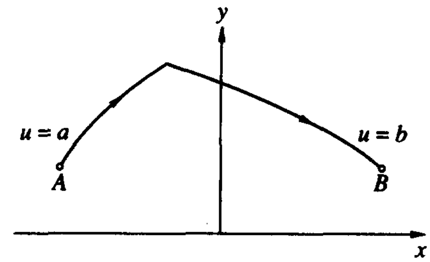

Fig. 2

$$L \in (-1/b, -1/(4b)) \cup (-1/(4a), -1/a) \tag{4.2}$$

A control corresponding to values of the parameter L in the interval $(-1/(4b), -1/(4a))$ will be called basic, as distinct from a singular control, which corresponds to values (4.2) of L.

The following propositions hold for a basic control.

1. The SC of problem (1.1), (1.2), (4.1) consists of two semiparabolas (Fig. 1): $x = e_a y^2, y > 0$, $e_a \in (1/(2b), 0)$ and $x = e_b y^2, y < 0, g \in (0, 1/(2b))$.

2. Rotating about the origin, an optimal trajectory reaches it in a finite time and in so doing crosses the SC a denumerable number of times. The time intervals between different switchings from $u = a$ to $u = b$ (or from $u = b$ to $u = a$) form a geometric progression.

3. The optimal control equals a to the right of the SC and b to the left of it.

Proof. It follows from the invariance of the problem with respect to the group (3.6) that the SC consists of a set of parabolas.

Let $L = 0$ and let $(x(t), y(t))$ be an admissible solution of system (1.1) with control $u(t)$ such that $y(t) > 0, t \in (t_1, t_2)$. Then the function $x(t)$ is monotonic and the curve $(x(t), y(t))$ may be represented by the equation $y = y(x)$. Consequently, functional (1.2) may be rewritten in the form

$$J = \int_{t_1}^{t_2} x^2 dt = \int_{x(t_1)}^{x(t_2)} x^2 \frac{dx}{y} \tag{4.3}$$

The greater $y = y(x)$, the smaller will be the interval on the right of equality (4.3). Let $A = (x(t_1), y(t_1))$, $B = (x(t_2), y(t_2))$ be the two end-points. A trajectory with a single switching (from $u = b$ to $u = a$) exists which lies above curves corresponding to any other admissible solutions that pass through the points A and B in the upper half-plane (Fig. 2). Thus if $(x(t), y(t))$ is an optimal trajectory, then there are either no switchings or there is just one: from b to a at $t \in (t_1, t_2)$. Analogous reasoning shows that when $y(t) < 0$ $(t \in (t_1, t_2))$ there are either no switchings or just one: from a to b.

Let $L \neq 0$. Let us assume that one of the branches of the SC is situated in the upper half-plane. Relations (2.2) and (2.3) must hold at its points. If the system is on that branch of the SC, the control $u = a$ or $u = b$ must be switched on and the motion will continue with that control until relation (2.2) holds again. Integrating system (2.4) with initial conditions (assuming, without loss of generality, that $y(0)$ is equal to 1)

$$y(0) = 1, \quad x(0) = hy^2(0), \quad p(0) = -x^2(0)/y(0), \quad q(0) = -Lx^2(0)$$

we deduce that the expression $q(t) + Lx^2(t)$ is a polynomial of degree 4 in t

$$f(t) = q(t) + Lx^2(t) = -\frac{1}{6}Lu^2t^4 - \frac{2}{3}Lut^3 + \frac{1}{12}ut^4 + \frac{1}{3}t^3 + ht^2 + h^2t - 2Lh^2 - Lt^2 - 2Lht$$

Investigation of the polynomial $f(t)$ for different choices of (u, L, h), carried out using the MAPLE system, has shown that in the case of negative u the polynomial has no real non-negative roots that do not exceed $1/u$; when $u > 0$, $f(t)$ has no real non-negative roots (in which case the control steers the system to infinity). Thus, there is at most one branch of the SC in the upper half-plane, and $u = a < 0$ to its right.

Analogous reasoning shows that there is at most one branch of the SC in the lower half-plane, and $u = b > 0$ to the left of it.

Since problem (1.1), (1.2), (4.1) possesses the following symmetry

$$L \to -L, \quad a \to -b, \quad b \to -a: V(x, y) \to V(-x, -y), \quad e_a \to -e_b, \quad e_b \to -e_a \tag{4.4}$$

it follows that the semiparabolas forming the SC are situated in quadrants symmetric about the origin.

It follows from the equation $\dot{x} = y$ that the motion is clockwise. If $e_a < 1/(2a)$ (resp. $e_b > 1/(2b)$), a trajectory starting on the upper (lower) part of the SC will never cross the SC again (Fig. 3a). If one of the coefficients e_c is zero, it follows from the symmetry relations (4.4) that the other coefficient must also vanish, and one has motion along a closed trajectory. Consequently, $1/(2a) \leqslant e < 0, 0 < g \leqslant 1/(2b)$. If $e = 1/(2a)$, the upper part of the SC is a trajectory of the system $\dot{x} = y, \dot{y} = u$ with $u = a$. As follows from Eqs (2.1), the value of the expression $q + Lx^2$ changes sign at each point of this trajectory; hence, on the trajectory itself, it must be true that $q + Lx^2 = 0$, and this must hold identically with respect to time as the system moves along the trajectory. Thus, motion along the parabola $x = y^2(2a)$ satisfies the Maximum Principle only when that parabola is a singular arc. Consequently, in basic control $e_a > 1/(2a)$. Analogous reasoning shows that $e_b < 1/(2b)$.

If the SC lies in the first and third quadrants, the corresponding trajectory recedes from zero (Fig. 3b). Thus, the semiparabolas forming the SC are situated in the second and fourth quadrants. We have thus proved propositions 1 and 3.

Let $C_1(e_a y_1^2, y_1)$, $C_2(e_b y_2^2, y_2)$, $C_3(e_a y_3^2, y_3)$, $C_4(e_a y_4^2, y_4)$ be consecutive switching points of the control. Then, by conditions (1.1), we have

$$\begin{aligned} y_2 &= y_1 + at_1, \quad e_b y_2^2 = e_a y_1^2 + y_1 t_1 + at_1^2/2 \\ y_3 &= y_2 + bt_2, \quad e_a y_3^2 = e_b y_2^2 + y_2 t_2 + bt_2^2/2 \end{aligned} \tag{4.5}$$

where t_1 is the time needed to go from point C_1 to point C_2, and t_2 is the time needed to go from point C_2 to point C_3. It follows from (4.5) that

$$\frac{y_2^2}{y_1^2} = g_a, \quad \frac{y_3^2}{y_2^2} = \frac{1}{g_b}; \quad g_c = \left[\frac{e_a - 1/(2c)}{e_b - 1/(2c)}\right]^{1/2} \tag{4.6}$$

from which it follows that

$$y_{n+2}/y_n = g_a/g_b < 1$$

and the set of switching points is thus shown to be denumerable

Using equalities (4.6), we obtain the relation

$$\frac{t_2}{t_1} = \frac{a}{b}\frac{y_3 - y_2}{y_2 - y_1} = -\frac{a}{b}\frac{1 + 1/g_b}{1 + 1/g_a} = \alpha \tag{4.7}$$

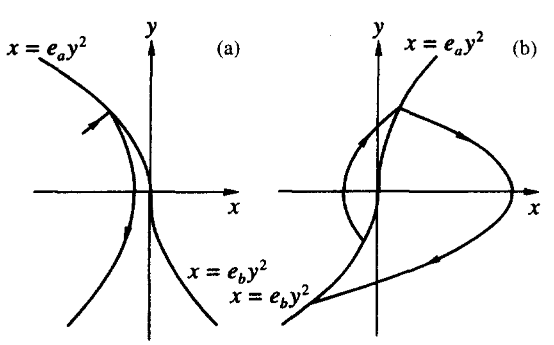

Fig. 3

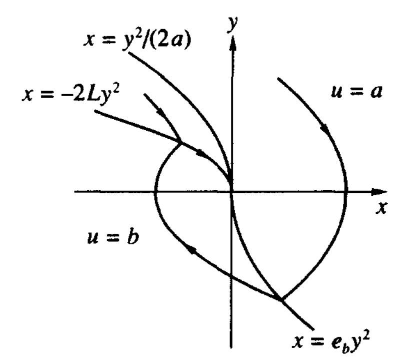

Fig. 4

Similarly, we obtain

$$\frac{t_3}{t_2} = -\frac{b}{a}\frac{1+g_a}{1+g_b} \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\frac{t_3}{t_1} = \frac{t_3 t_2}{t_2 t_1} = q < 1, \quad e_a \in \left(\frac{1}{2a}, 0\right), \quad e_b \in \left(0, \frac{1}{2b}\right) \tag{4.9}$$

Relation (4.9) is a special case of the general relation

$$t_{n+2}/t_n = q < 1 \tag{4.10}$$

Thus, the time necessary to reach the origin is the sum of the geometric progression

$$T = t_1(1+\alpha)(1+q+q^2+\ldots) = t_1(1+\alpha)/(1-q) \tag{4.11}$$

We have thus proved Proposition 2.

When $L \in (-1/b, -1/(4b)) \cup (-1/(4a), -1/a)$, the control has two switchings and a singular arc of first order. The nature of the optimal synthesis for $L \in (-1/(4a), -1/a)$ is shown in Fig. 4.

The rigorous basis of these control laws is obtained via dynamic programming. The structure of Bellman's function, which depends on two constants, was obtained in Section 3 (see (3.13)). It is still necessary to determine the constants A_c and the coefficients e_c of the semiparabolas that make up the switching curve. This will constitute a complete solution of the problem. In the computation of A_c and e_c, the parameters a, b and L are assumed to be given. Generally speaking, the volume of computations may be reduced by virtue of symmetry relations (4.4).

The following system of four equations is considered in the interval $L \in (-1/(4b), -1(4a))$

$$V^a(x, y) = V^b(x, y), \quad V_y^c + Ly^2 = 0 \quad (x = e_c y^2) \tag{4.12}$$

When $x = e_c y^2$ is substituted into Eqs (4.12), a common factor y^4 or y^5 appears, and its cancellation leads to the following equalities in terms of A_c, e_c and a, b, L

$$\begin{aligned} A_a\left|\frac{1}{2a} - e_c\right|^{5/2} + \omega_c\left(\frac{1}{a} + L\right)\left(e_c^2 - \frac{2}{3a}e_c + \frac{2}{15a^2}\right) = \\ = A_b\left|\frac{1}{2b} - e_c\right|^{5/2} + \omega_c\left(\frac{1}{b} + L\right)\left(e_c^2 - \frac{2}{3b}e_c + \frac{2}{15b^2}\right) \end{aligned} \tag{4.13}$$

$$\frac{5}{2a}\omega_c A_c\left|\frac{1}{2c} - e\right|^{3/2}\operatorname{sign}\left(\frac{1}{2c} - e\right) - \left(\frac{1}{a} + L\right)\left(e_c^2 - \frac{2}{a}e_c + \frac{2}{3a^2}\right) + Le_c^2 = 0 \tag{4.14}$$

where $\omega_a = -1$, $\omega_b = 1$

These equalities form a system of transcendental equations is unknowns A_c and e_c, which have been solved numerically, using the MAPLE system, for specific values of a, b and L.

For $L \geqslant -1/(4a)$ and $L \leqslant -1/(4b)$, Eq. (2.6) gives $e_c = -2L$. Substituting this value into Eq. (4.14), we obtain the following value for the constant $A_c = A_c^*$

$$A_c^* = -\frac{8\sqrt{2}}{15}\omega_c\frac{(3Lc+1)}{\sqrt{c(4Lc+1)}} \tag{4.15}$$

If $L \geqslant -1/(4c)$, the function $A_a^*(x, y)$, equal to $V^a(x, y)$ as in (3.13) with the constant (4.15), defines Bellman's function in the part of the domain N^a between the semiparabolas $x = -2Ly^2$ and $x = y^2/(2a)$. In the other part of N^a, the function $V^a(x, y)$ is defined with a constant A_a which, together with A_b and e_b, must be sought using a system of three equations, including (4.13) with $c = b$, (4.14) with $c = b$, and the modification of Eq. (4.13) with $c = a$, in which we substitute $A_a = A_a^*$ and $e_a = -2L$. Note that if $L = -1/(4a)$, then $e = 1/(2a)$, and to find A_a, A_b and e_b we use the system consisting of (4.13) with $c = a$, (4.13) with $c = b$, and (4.14) with $c = b$. There is no longer any need for the constant A_a^*, since the curve $x = -2Ly^2$ coincides with the curve $x = y^2/(2a)$. Relation (4.14) with $c = a$ then becomes an

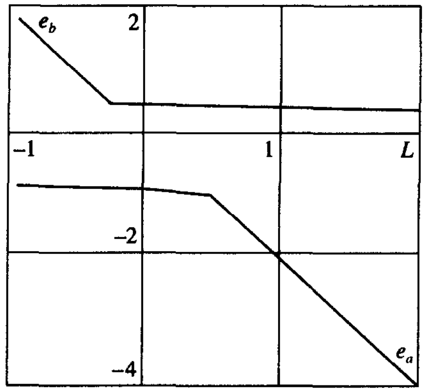

Fig. 5

identity. While $A_a^* \to \infty$ as $L \to -1/(4a,) + 0$, the quantity $V_a^*(x, y)$ tends to a finite limit for $x = y^2/(2a)$:

$$V_a^* = -3y^5/(80a^3)$$

The results of the computations for $a = -1/2$, $b = 1$, are as follows:

L	-1.0	-0.5	-0.2	0	0.2	0.5	1.0	2.0
A	1.53960	1.31993	1.17206	1.06959	0.96668	0.81200	0.55378	0.03657
A^*	–	–	–	–	–	–	0.53333	1.23168
B	0.17893	0.48594	0.66907	0.79046	0.91098	1.08866	1.34924	1.75991
B^*	0.87093	0.37712	–	–	–	–	–	–
e_a	−0.82684	−0.84953	−0.86990	−0.88897	−0.91610	−1.00000	−2.00000	−4.00000
e_b	2.00000	1.00000	0.48047	0.44510	0.42799	0.41342	0.40000	0.38632

and are illustrated in Fig. 5.

5. CONSTRAINS OF THE SAME SIGN

Let us now consider the problem of minimizing the family of functionals (1.2) for motions of system (1.1) in the case when the range of admissible values of the control parameter does not contain the origin

$$ab > 0; \quad a > 0: \quad L \geq -1/b; \quad b < 0: \quad L \leq -1/a \tag{5.1}$$

In that case the controllability domain (the domain of initial values from which admissible controls will steer the system to the terminal set) is not the whole (x, y) plane: starting from a point (x^0, y^0), the system may be brought to the origin only provided that

$$y^{0^2}/(2b) \leq x^0 \leq y^{0^2}/(2a); \quad a > 0: \quad y^0 < 0; \quad b < 0: \quad y^2 > 0 \tag{5.2}$$

The controllability domain for the case $a > 0$ is bounded by the upper semiparabola ($x = y^2/(2a)$) and the lower semiparabola ($x = y^2/(2b)$) (the dashed curves in Fig. 6).

Everywhere henceforth we shall assume that the constraints imposed on the control have positive signs. Consideration of the case in which the range of admissible control values lies entirely on the negative semi-axis is analogous. If the system is outside the limits of the controllability domain, no admissible control will steer it to the origin. Note that when the system is on the boundary of the controllability domain, say the upper (lower) semiparabola, the only control that does not take it outside the limits of the controllability domain is motion along the semiparabola in question with control $u = a$ (resp. $u = b$). This control law will steer the system to the origin. All other types of control will take it out of the controllability regime.

By what was stated in Section 2, we have the following control values satisfying the Maximum Principle

$$u = \begin{cases} a & \text{if} \quad q + Lx^2 > 0 \\ b & \text{if} \quad q + Lx^2 < 0 \\ -1/(4L) & \text{if} \quad q + Lx^2 \equiv 0, \quad t \in (t_1, t_2) \end{cases} \tag{5.3}$$

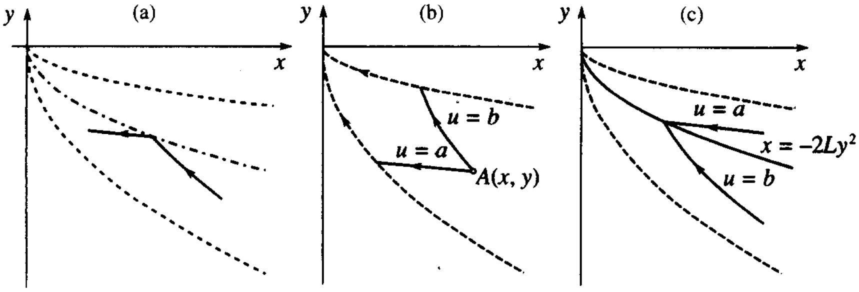

Fig. 6

The last value corresponds to motion along the singular arc (2.6). By virtue of the constraints on the control, the corresponding control will be possible only if

$$-1/(4a) \le L \le -1/(4b) \tag{5.4}$$

Constraints (5.4) have a simple geometrical interpretation: as soon as the relation between the parameters of the problem becomes such that the semiparabola $x = -2Ly^2$, $y < 0$, is inside the controllability domain, one has a control with a first-order singular arc.

Note that, besides the singular arc, there is no SC inside the controllability domain. Indeed, suppose the contrary: suppose that there is a SC in the interior of the controllability domain at which the control changes value from $u = 1$ ($u = b$) to $u = b$ ($u = a$). This SC must separate two domains with positive and negative values of $q + Ly^2$. Suppose in the domain where $q + Ly^2 > 0$ a motion with control $u = a$ has begun (see formula (5.3)). On reaching the SC and changing the control value to $u = b$, the system continues to move in the same domain in which it began (see Fig. 6a, where the SC is the dash–dot curve). Hence the value of the control should not change. This contradiction proves that the only possible SC is the singular arc (where the control changes value from $u = a$ or $u = b$ to $u = -1/(4L)$).

Thus, for $L \in (-1/b, -1/(4b)) \cup (-1/(4a), \infty)$ two control laws using the extreme values of the control turn out to be possible (see Fig. 6b): motion with control $u = a$ ($u = b$) up to the lower (upper) boundary of the controllability domain $x = y^2/(2b)$ ($x = y^2/(2a)$), and then motion along the boundary with control $u = b$ ($u = a$).

The question as to the optimality of these regimes is solved using Bellman's function. At any point $A(x, y)$ on the boundary of the controllability domain, the value of Bellman's function may be found by direct integration of functional (1.2) along the relevant part of the boundary (since any encounter with the boundary implies going outside the controllability domain). Having integrated (1.2) along the semiparabolas $x = y^2/(2a)$, $y < 0$ and $x = y^2/(2b)$, $y < 0$, we obtain the following values of Bellman's function on the lower boundary (superscript minus) and upper boundary (superscript plus)

$$V^-(y) = V\left(\frac{1}{2b}y^2, y\right) = -\frac{1}{20}\frac{Lb+1}{b^3}y^5, \quad V^+(y) = V\left(\frac{1}{2a}y^2, y\right) = -\frac{1}{20}\frac{La+1}{a^3}y^5 \tag{5.5}$$

Using the first (second) of these relations, we can determine the value of the constant A_a (A_b) in expression (3.13). We have

$$\begin{aligned} A_a &= -\frac{1}{15}\frac{\sqrt{2}(La+1)(8b^2-12ab)+3a^2\sqrt{2}}{(b-a)\sqrt{ab(b-a)}} \\ A_b &= \frac{1}{15}\frac{\sqrt{2}(Lb+1)(8a^2-12ab)+3b^2\sqrt{2}}{(b-a)\sqrt{ab(b-a)}} \end{aligned} \tag{5.6}$$

The signs of the expressions

$$W^c = V_y^c + Lx^2 \tag{5.7}$$

were used to establish the character of the synthesis. On parabolas of the form $x = ry^2$, $y < 0$, formulae (5.7) may be expressed as functions of the variables y and r

$$W^c = \alpha_c(r)y^4 \tag{5.8}$$

Thus, to investigate the signs of formulae (5.7), it will suffice to investigate the signs of the functions $\alpha_c(r)$, $r \in (1/(2b), 1/(2a))$.

It has been shown that for $L < -1/(4a)$ ($L > -1/(4b)$) the domains of values of $\alpha_c(r)$ lie entirely on the negative (positive) real semi-axis. Hence it follows that the values of formula (5.7) are negative (positive) at all interior points of the controllability domain, whence, in view of (5.3), we deduce that for $L < -1/(4a)$ ($L > -1/(4b)$) an optimal control law consists of the following elements: start at any point $A(x, y)$ of the controllability domain, necessarily with control $u = b$ ($u = a$) and, after reaching the upper (lower) boundary of the controllability domain, move along it to the origin, using the control $u = a$ ($u = b$). This regime is represented by the upper (lower) trajectory in Fig. 6b.

In the range of L values (5.4), the domains of values of the function $\alpha_c(t)$ include zero and, besides the ranges of r values where they have the same signs, intervals exist at whose points the function $\alpha_c(r)$ have different signs. This dictates the assumption that, when condition (5.4) holds, the optimal control must include an interval of motion along the singular arc (2.6) (see Fig. 6c). The values of the constants A_c in formula (3.13) are determined from the values of Bellman's function on the singular arc. The values of Bellman's function at points of the singular arc are determined by direct integration of functional (1.2) along the singular arc (2.6)

$$V^s(y) = V(-2Ly^2, y) = \frac{12}{5}L^3y^5 \tag{5.9}$$

Along the singular arc it must be true that

$$V^c(-2Ly^2, y) = \frac{12}{5}L^3y^5 \tag{5.10}$$

It follows from (3.13) and (5.10) that

$$A_c = -\frac{8}{15}\sqrt{\frac{2}{|c|}}\frac{(3Lc+1)}{\sqrt{|1+4Lc|}} \tag{5.11}$$

It has been shown that the following relations hold in the domain above the singular arc (see Fig. 6c)

$$V_y^b + Lx^2 < 0, \quad x > -2Ly^2; \quad V_y^b + Lx^2 = 0, \quad x = -2Ly^2 \tag{5.12}$$

while in the domain beneath the singular arc

$$V_y^a + Lx^2 > 0, \quad x < -2Ly^2; \quad V_y^a + Lx^2 = 0, \quad x = -2Ly^2 \tag{5.13}$$

Relations (5.3), (5.12) and (5.13) imply that the synthesis illustrated in Fig. 6 is optimal

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REFERENCES

1. ZELIKIN, M. I. and BORISOV, V. F., *Theory of Chattering Control, with Applications to Astronautics, Economics and Engineering*. Birkhäuser, Boston, 1994.
2. FULLER, A. T., Study of an optimum non-linear control system. *J. Electronics and Control*, 1963, **15**, 1, 63–71.
3. WONHAM, W. M., Note on a problem in optimal non-linear control. *J. Electronics and Control*, 1963, **15**, 1, 59–62.
4. ALDAKIMOV, Yu. V., MELIKYAN, A. A. and NAUMOV, G. V., Mode reconstruction in a one-parameter family of optimal control problems. *Prikl. Mat. Mekh.*, 2001, **65**, 3, 400–407.
5. GABASOV, R. and KIRILLOVA, F. M., *Singular Optimal Controls*. Nauka, Moscow, 1973.

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